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Stochastic responses of vibro-impact duffing oscillator excited by additive Gaussian noise

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Abstract

The stationary responses of vibro-impact Duffing oscillator excited by additive Gaussian white noise are studied by using the quasi-conservative averaging method. With the help of a non-smooth variable transformation and the Dirac delta function, the response probability density functions (PDFs) are formulated analytically. Meanwhile, the results are validated numerically using Monte Carlo simulations.

1. Introduction

Many non-smooth factors arise very naturally in engineering applications [1,2], such as impacts, collisions, dry frictions and so on. In the last few years, a considerable amount of research activity has focused on non-smooth dynamical systems. Using a local Pioncaré map, Shaw and Holmes [3], Nordmark [4–6], Luo [7–9] and Weger et al. [10,11] explored the bifurcation of piecewise linear systems and vibration systems with constraints, respectively. Feng [12,13] explored the mean response of non-smooth systems by using the mean Poincaré map. Huang et al. [14] obtained the stationary responses of a multi-degree-of-freedom vibro-impact system under white noise excitations according to the Hertz contact law. An important contribution into theory of vibration systems with impact had been made by Zhuravlev [15], who proposed a non-smooth variable transformation to deal with non-smooth characteristics, based on which, recently, Dimentberg [16–18] and Iourtchenko [19,20] explored impact energy losses for linear vibration systems with impact and the response probability density functions (PDFs) of stochastic linear systems with impact numerically. Namachchivaya [21,22] developed an averaging approach to study the dynamic actions of a vibro-impact system excited by random perturbations.

Non-smooth dynamics with impacts are always trouble with their inherent difficult of analytical study. The difficulties depend on the additional finite relations between impacts and rebound velocities. On one hand, the impact instants are not known in advance, but are rather state-dependent, i.e., governed by the equations of motion. So it is reasonable that the impact instants are depicted by constraints which depend on the state of

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systems. On the other hand, the impact instants govern the corresponding velocity jumps, i.e., the differential equations of motion on the impact instants are complemented by impact conditions. The previous literatures fastened on the study of either nonlinear smooth systems or non-smooth systems which vector field itself is linear. Particularly, it is an interesting work to consider the responses of non-smooth nonlinear systems driven by random noise.

This paper is organized as follows. In Section 2, the transformed system is obtained by using a non-smooth transformation and the Dirac delta function. In Section 3, the quasi-conservative averaging method [23,24] is used to deal with the corresponding transformed smooth system with Dirac delta function. In Section 4, numerical simulations are also used to verify analytical results. A conclusion is presented in the last section.

2. The non-smooth transformation

Consider Duffing system with impacts, the corresponding differential equation of motion is as follows:

$$\ddot{x} + ax + b\dot{x} + cx^3 = f\xi(t), \quad x > 0, \tag{1}$$

$$\dot{x}_+ = -r\dot{x}_-, \quad x = 0, \tag{2}$$

where the constraint equation is

$$h(x, \dot{x}) = x = 0 \tag{3}$$

and the impact condition is

$$\dot{x}_+ = -r\dot{x}_-, \quad \text{where } 0 < r \leqslant 1. \tag{4}$$

$\xi(t)$ is a Gaussian white noise satisfied

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t+\tau)\xi(t) \rangle = \delta(\tau).$$

Here a, b, c, f are constants, r is the restitution coefficient, $\dot{x}_+$, $\dot{x}_-$ are the velocities of system after and before impacts, respectively. Motion of system is "freedom" until the constraint Eq. (3) is satisfied and the impact condition (4) is then imposed. The instant of impact t_* corresponding to $x = 0$, namely, $x(t_*) = 0$, is not known in advance. Obviously, when value of $(1-r)$ approaches to zero, the impact losses of system becomes very small, this means that the system is quasi-conservative one and the quasi-conservative averaging method is valid here. Firstly, a non-smooth transformation of state variables is introduced

$$x = x_1 = |y|, \quad \dot{x} = x_2 = \dot{y}\,\mathrm{sgn}y, \quad \ddot{x} = \ddot{y}\,\mathrm{sgn}y, \tag{5}$$

$$\mathrm{sgn}y = \begin{cases} 1, & y > 0, \\ -1, & y < 0. \end{cases}$$

When Eq. (5) is introduced in Eqs. (1) and (2), we can obtain, respectively,

$$\ddot{y} + ay + b\dot{y} + cy^3 = f\xi(t)\mathrm{sgn}(y), \quad t \neq t_*, \tag{6}$$

$$\dot{y}_+ = r\dot{y}_-, \quad t = t_*. \tag{7}$$

Obviously, the jump of the transformed velocity becomes proportional to $1-r$ instead of $1+r$ for the original argument x. Assume that only one impact present in a period of system. It is reasonable to consider Eq. (7) as an additional impulsive damping of Eq. (6) since the jump of the velocity only appears at the instants of impacts, the additional term $(\dot{y}_- - \dot{y}_+)\delta(t - t_*)$ can be obtained from Eq. (7) by using the Dirac delta function $\delta(y)$. Using the following relations:

$$\delta(t - t_*) = |\dot{y}|\delta(y), \quad (\dot{y}_- - \dot{y}_+)\delta(t - t_*) = (1 - r)\dot{y}|\dot{y}|\delta(y). \tag{8}$$

Eqs. (6) and (7) can be incorporated into

$$\ddot{y} + ay + b\dot{y} + cy^3 = f\xi(t)\mathrm{sgn}(y) - (1 - r)\dot{y}|\dot{y}|\delta(y). \tag{9}$$

Thus, the original system with impact was reduced to one without impact expressed by Eq. (9). The second term $(1-r)\dot{y}|\dot{y}|\delta(y)$ on the right-hand side of Eq. (9) can be regarded as an impulsive damping term. When the coefficients of all excited terms are proportional to small parameter, the transformed system permits strict analytical study by some asymptotic methods of averaging over a period.

3. Quasi-conservative averaging

Now we consider the transformed system (9) by using the quasi-conservative averaging method as long as the different value between the input energy of random excitation and the loss energy of damping is smaller than the total energy of the system. Firstly, Eq. (9) is equivalent to a pair of first-order equations:

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -ay_1 - by_2 - cy_1^3 + f\xi(t)\mathrm{sgn}(y_1) - (1-r)y_2|y_2|\delta(y_1). \end{aligned} \tag{10}$$

This can be converted to the Itô-type stochastic differential equations as follows:

$$\begin{aligned} \mathrm{d}y_1 &= y_2\,\mathrm{d}t, \\ \mathrm{d}y_2 &= [-ay_1 - by_2 - cy_1^3 - (1-r)y_2|y_2|\delta(y_1)]\,\mathrm{d}t + f\,\mathrm{sgn}(y_1)\,\mathrm{d}W(t). \end{aligned} \tag{11}$$

The corresponding total energy function and the potential function of system are, respectively,

$$H = \frac{y_2^2}{2} + U(y_1), \tag{12}$$

$$U(y_1) = \frac{a}{2}y_1^2 + \frac{c}{4}y_1^4. \tag{13}$$

Inserting Eq. (12) with respect to t, and combining the result with Eq. (10), we obtain

$$\dot{H} = -(1-r)y_2^2|y_2|\delta(y_1) - by_2^2 + f\xi(t)y_2\,\mathrm{sgn}(y_1). \tag{14}$$

Obviously, When the parameters $(1-r)$, b and f are small, y_1 and y_2 are two fast varying random process while H is a slowly varying random process, which may be approximated by a Markovian process governed by the Itô stochastic differential equation

$$\mathrm{d}H = m(H)\,\mathrm{d}t + \sigma(H)\,\mathrm{d}W(t), \tag{15}$$

where the square of drift coefficients is

$$m(H) = -(1-r)y_2^2|y_2|\delta(y_1) - by_2^2 + \tfrac{1}{2}f^2 \tag{16}$$

and the diffusion coefficients is

$$\sigma^2(H) = f^2y_2^2. \tag{17}$$

The corresponding mean equation of Eq. (9) can be obtained by using the quasi-conservative averaging procedure. It is important to note that the impulsive damping term should be averaged over a half-period since there are two impacts in each period. Averaging the drift and diffusion coefficients, we obtain

$$\begin{aligned} \overline{m}(H) = \langle m(H)\rangle_t &= \frac{1}{T(H)}\int_{-A}^{A} \frac{[m(H)]_{y_2=\sqrt{2H-2U(y_1)}} + [m(H)]_{y_2=-\sqrt{2H-2U(y_1)}}}{\sqrt{2H-2U(y_1)}}\,\mathrm{d}y_1 \\ &= \frac{1}{T_{1/4}(H)}\left(-S(H) - bR(H) + \frac{1}{2}f^2T_{1/4}(H)\right), \end{aligned} \tag{18}$$

$$\overline{\sigma}^2(H) = \langle \sigma^2(H) \rangle_t = \frac{1}{T(H)} \int_{-A}^{A} \frac{[\sigma^2(H)]_{y_2=\sqrt{2H-2U(y_1)}} + [\sigma^2(H)]_{y_2=-\sqrt{2H-2U(y_1)}}}{\sqrt{2H-2U(y_1)}} \, \mathrm{d}y_1$$
$$= \frac{f^2}{T_{1/4}(H)} R(H), \tag{19}$$

where

$$S(H) = (1-r)H, \tag{20}$$

$$R(H) = \int_0^A \sqrt{2H - 2U(y_1)} \, \mathrm{d}y_1 \text{ and } \frac{\partial R}{\partial H} = T_{1/4}, \tag{21}$$

$$\frac{T(H)}{4} = T_{1/4}(H) = \frac{1}{\sqrt{a + cA^2}} F\left(K, \frac{\pi}{2}\right), \tag{22}$$

where A is the positive root of the following equation:

$$U(A) = H. \tag{23}$$

$F(K, \pi/2)$ is the complete elliptic integral of the first kind, which is defined as

$$F\left(K, \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - K^2 \sin^2 \theta}} \, \mathrm{d}\theta. \tag{24}$$

And K is its modulus, in the present case

$$K = \frac{A\sqrt{c}}{\sqrt{2a + 2cA^2}}. \tag{25}$$

We obtain the corresponding mean Itô differential equation of Eq. (15) instead of the drift coefficients and the diffusion coefficients as follows:

$$\mathrm{d}H = \overline{m}(H)\,\mathrm{d}t + \overline{\sigma}(H)\,\mathrm{d}W(t).$$

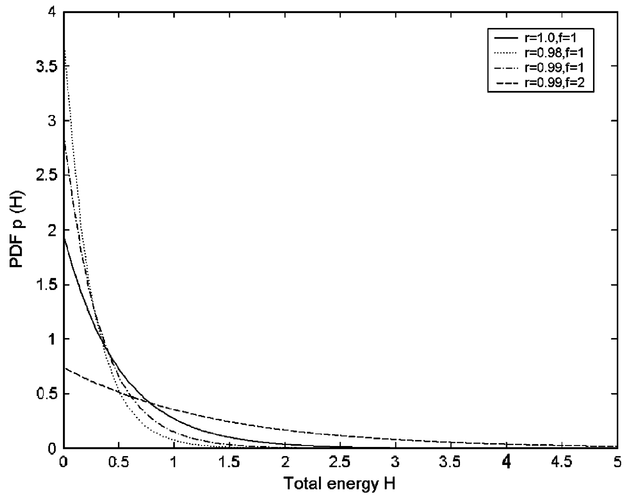

Fig. 1. Stationary PDFs of the total energy $p(H)$.

The corresponding mean FPK equation of above equation is

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial H}[\overline{m}(H)p] + \frac{1}{2}\frac{\partial^2}{\partial H^2}[\overline{\sigma}^2(H)p]. \tag{26}$$

The exact solution of the mean FPK Eq. (26) is desired but very difficult to obtain by far because of nonlinear factors. We seek the stationary solution, namely, $\partial p/\partial t = 0$ in Eq. (26). Then, we can obtain

$$p(H) = C\,\mathrm{e}^{-\lambda(H)}. \tag{27}$$

When the boundary conditions are

$$\begin{cases} 0 \leqslant p < \infty, & H = 0, \\ p \to 0, \dfrac{\mathrm{d}p}{\mathrm{d}H} \to 0, & H \to \infty, \end{cases} \tag{28}$$

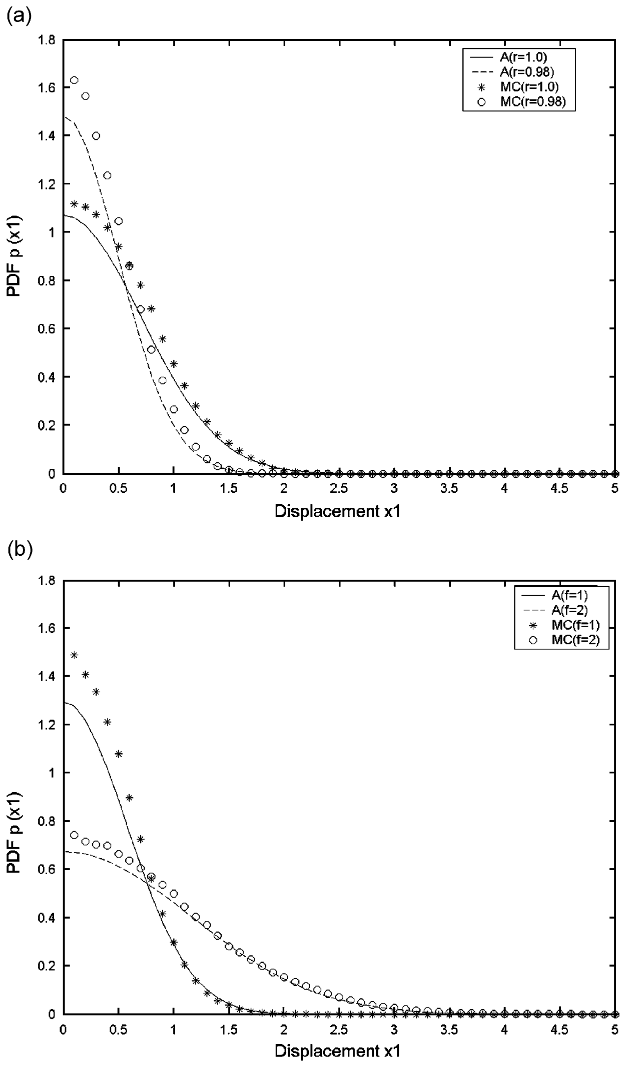

Fig. 2. Stationary PDFs of the displacement $p_{X_1}(x_1)$: (a) $f = 1$ and (b) $r = 0.99$.

where the symbol '→' means the limit operation. In Eq. (27) C is the normalization constant and $\lambda(H)$ is as follows:

$$\lambda(H) = \frac{2bH}{f^2} + \frac{21}{f^2}\int_0^H \frac{S(x)}{R(x)}\,\mathrm{d}x - \ln\frac{T_{1/4}(H)}{T_{1/4}(0)}. \tag{29}$$

The stationary unite PDF of variable y_1 and y_2 can be obtained easily by using Eqs. (12) and (13)

$$p_{Y_1Y_2}(y_1, y_2) = \left.\frac{p(H)}{T(H)}\right|_{H=(y_2^2/2)+(a/2)y_1^2+(c/4)y_1^4}. \tag{30}$$

By using the transformation formula (5), the stationary unite PDF of the original variable x_1 and x_2 can be obtained

$$\bar{p}(x_1, x_2) = p_{Y_1Y_2}(x_1, x_2) + p_{Y_1Y_2}(-x_1, -x_2). \tag{31}$$

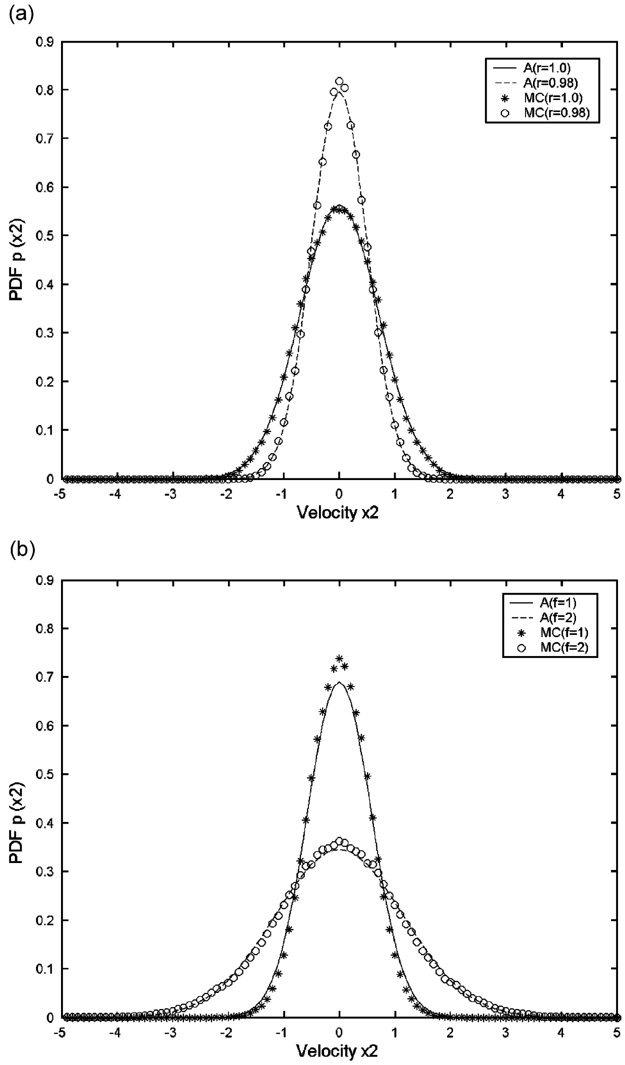

Fig. 3. Stationary PDFs of the velocity $p_{X_2}(x_2)$: (a) $f = 1$ and (b) $r = 0.99$.

We know that the PDF $p_{Y_1Y_2}(y_1, y_2)$ is symmetrical from Eqs. (12), (13) and (30), the PDF $\bar{p}(x_1, x_2)$ can be rewritten as

$$\bar{p}(x_1, x_2) = 2p_{Y_1Y_2}(x_1, x_2). \quad x_1 \geqslant 0. \tag{32}$$

The corresponding PDF of variable x_1 and x_2 is as follows, respectively,

$$p_{X_1}(x_1) = \int_R s\,\bar{p}(x_1, s)\,\mathrm{d}s, \tag{33}$$

$$p_{X_2}(x_2) = \int_{R^+} s\,\bar{p}(s, x_2)\,\mathrm{d}s. \tag{34}$$

4. Numerical results

In this section, we compare the analytical(A) results in Section 3 with numerical simulations by Monte Carlo (MC) methods which is based of the original system denoted by Eqs. (1) and (2). Here, we set parameters εb, εc, $\varepsilon^{1/2} f$ instead of b, c, f, respectively, since the averaging method is only effective for small perturbations.

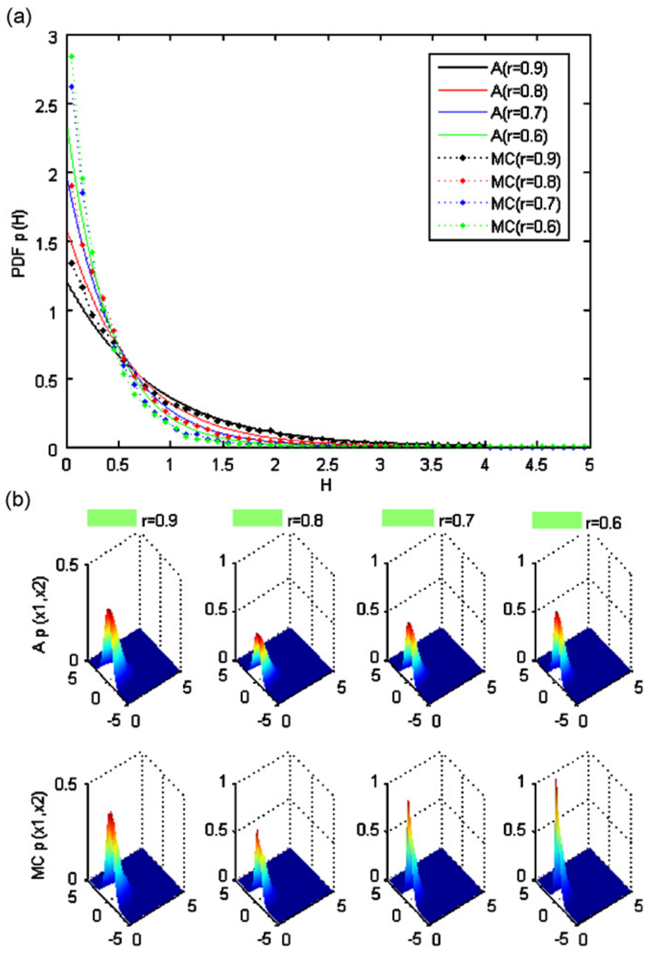

Fig. 4. Analytical and numerical results for different values of the restitution coefficient: (a) stationary PDFs of the total energy $p(H)$ and (b) unite stationary PDFs $\bar{p}(x_1, x_2)$.

Suppose $a = 1$, $b = 1$, $c = 1$, $\varepsilon = 0.01$ in Figs. 1–3. If the noise intensities f are small and the restitution coefficient r is close to 1, the analytical results agree with the numerical simulations. Fig. 1 demonstrate the stationary PDFs of the total energy in formula (27) for different values of the restitution coefficient and noise intensities, it is clear that the increase of the restitution coefficient r or noise intensities f can both lead to the total energy responses more flat, namely, decrease in the probability that smaller energy of system.

For different values of the restitution coefficient r and noise intensities f, the stationary displacement responses are displayed in Fig. 2, while the stationary velocity responses are shown in Fig. 3, respectively. Where, the analytical results are based on formula (33) and (34). Firstly, more close to 1 for the restitution coefficient r, lower peak of the displacement PDFs in Fig. 2(a) and the velocity PDFs in Fig. 3(a), while the increase of noisy intensity f lead to decrease of the displacement PDFs in Fig. 2(b) and the velocity PDFs in Fig. 3(b). Secondly, it can be seen that the displacement PDFs's peaks appear on place of the constraints, and the velocity PDFs are symmetry and smooth. It is obvious that the approximate analytical results are close to the numerical ones in Fig. 3. Fig. 2 reflects that the image is almost identical between analytical results and numerical simulations.

Let the scale parameter $\varepsilon = 0.1$, that is $a = 1$, $\varepsilon b = 0.1$, $\varepsilon c = 0.1$, $\varepsilon^{1/2} f = 0.5$. When the restitution coefficient r is small corresponding to the inelastic impact, Fig. 4 provide the stationary PDFs of the total energy and the unite PDFs of displacement and velocity for different values r. In Fig. 4(a) it is clear that the PDFs's peaks in Monte Carlo simulation results are higher than ones in the analytical results as the decrease of the restitution coefficient r, more details are shown in Fig. 4(b). The problem should be mentioned. On one hand, the smaller restitution coefficient r leads to the more impact energy losses, the larger noise intensity f should be required to provide the impact, which may result in the inaccuracy in the approximate averaging method. On the other hand, when we make simulations, the larger noise intensity f increases numerical difficulty in tracing the exact instants of impacts, as a result, additional points near the constraints lead to higher peaks. The stationary unite PDFs in Fig. 4(b) are similar to the truncated ones in the situation without impact. These results resemble the conclusions as expected for the linear systems.

5. Conclusions

To our knowledge, the study on dynamical systems of linear non-smooth has many valuable results. In the paper, the effective formulas are presented to investigate the stationary PDFs of responses of a single-degree-of-freedom vibro-impact oscillator under additive white noise excitation. It has been demonstrated that the stationary PDFs of the total energy, the displacement, the velocity and the unite PDFs appear the peak near the constraint place, the effects of the restitution coefficient and noisy intensities are shows numerically. According to the P-bifurcation concept, no bifurcation occurs in the considered system.

In this paper, the procedures are extensive for single-degree-of-freedom vibro-impact systems and provide leads for exploring multi-degree-of-freedom vibro-impact systems under stochastic excitation, which appeals to appearance of the extensive non-smooth transformation.

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